DEPARTMENT OF MATHEMATICS, AHMADU BELLO UNIVERSITY, ZARIA.

FIRST SEMESTER EXAMINATIONS 2024/2025

MATH309- Analytical Dynamics Time allowed: 3hr.

Instruction: Answer any four(4) Questions

1. (a) A particle starts from the origin and the components of its velocity parallel to the axes of coordinates at time t are $2t + 3$ and $4t$. Find the path and the position of the particle when it is moving in direction equally inclined to the axes

   (b) Four uniform rods are freely joined at their extremities and formed a parallelogram ABCD which is suspended by the joint A and is kept in shape by a string AC. Prove that the tension on the string is equal to half the weight of the four rods.

2. (a) Obtain the tangential component of velocity of a particle moving along a plane curve.

   (b) If the radial and transverse velocities of a particle are always proportional to each other, show that the equation to the path is equiangular spiral.

3. (a) Obtain the rotational kinetic energy of a rigid body.

   (b) A cube with one corner fixed is moving under no external forces, if $\omega_1, \omega_2, \omega_3$ are the angular velocities about the three edges meeting at a fixed corner, prove that $\omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2$ is constant.

4. Obtain the general expression for the kinetic energy of a dynamic system

5. (a) A bead slides on a wire in the shape of a cycloid described by equations $x = a(\theta - sin\theta), y = a(1 + cos\theta), 0 \leq \theta \leq 2\pi$. Calculate its kinetic energy.

   (b) Using Lagrange's equation, obtain an equation of motion for a simple pendulum.

6. State and prove the conservation of angular momentum of a system of particles.

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1a) Given $\dot{x} = 2t + 3$ and $\dot{y} = 4t$

$\Rightarrow x = \int \dot{x}\,dt = \int (2t+3)\,dt = t^2 + 3t + C_1$ and,

$y = \int \dot{y}\,dt = \int 4t\,dt = 2t^2 + C_2$

Since the particle starts from the origin $\Rightarrow x_0 = 0$ and $y_0 = 0, t_0 = 0$

Hence $C_1 = 0$ and $C_2 = 0$ by substitution. $\Rightarrow y = 2t^2$ and $x = t^2 + 3t$

Making t the subject of equation in y, we have;

$$t = \sqrt{\frac{y}{2}} \quad \text{———} (1)$$

By substituting eqn (1) into the value of x, we have

$$x = \left(\sqrt{\frac{y}{2}}\right)^2 + 3\sqrt{\frac{y}{2}} = \frac{y}{2} + 3\sqrt{\frac{y}{2}}$$

$$x - \frac{y}{2} = 3\sqrt{\frac{y}{2}} \Rightarrow \frac{2x - y}{2} = 3\sqrt{\frac{y}{2}} \Rightarrow \left(\frac{2x-y}{6}\right)^2 = \left(\sqrt{\frac{y}{2}}\right)^2$$

$$\frac{(2x-y)^2}{36} = \frac{y}{2} \Rightarrow \frac{4x^2 - 4xy + y^2}{18} = y$$

$$\Rightarrow x^2 - xy + \frac{y^2}{4} = \frac{9y}{2} \Rightarrow 4x^2 - 4xy - 18y + y^2 = 0$$

Which is the required equation (path) of the particle which represent a Parabola.

Since the position of the particle is moving in direction equally inclined to the axis $\Rightarrow \frac{dy}{dt} = \pm \frac{dx}{dt}$ i.e $\dot{y} = \pm \dot{x}$

$$4t = \pm(2t+3) \Rightarrow 4t = 2t + 3 \text{ or } 4t = -(2t+3)$$

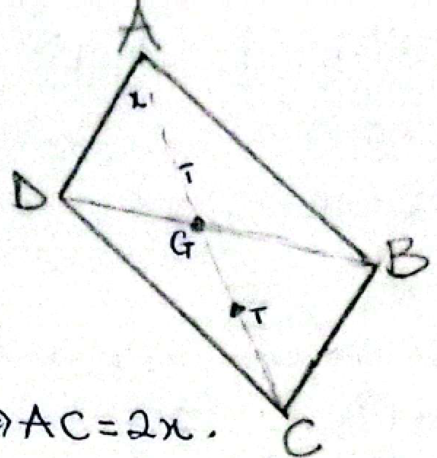
$$\Rightarrow 2t = 3 \text{ or } 6t = -3 \Rightarrow t = 3/2 \text{ or } t = -1/2$$

Hence $t = 3/2$ (positively) which is the position of the particles when moving in direction equally inclined to the axis.

1b) Proof:

Let T be the tension in the string AC and G be the point of intersection of the diagonal AC and BD respectively.

Let $4w$ be the weight of the four rods which may be taken to act at G. Here, A is the fixed point.



Then, Let $AG = x \Rightarrow AC = 2x$.
Let the system be slightly displaced about the vertical AG, so that x becomes $x + \partial x$. Then the equation of the virtual work is;

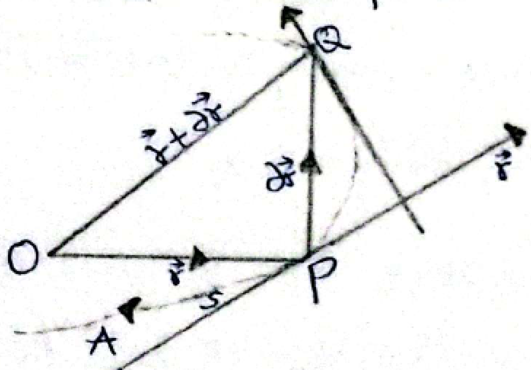
$$4w\partial(AG) - T\partial(AC) = 0 \Rightarrow 4w\partial x - T\partial(2x) = 0$$

$$\Rightarrow (4w - 2T)\partial x = 0 \text{ But Since } \partial x \neq 0 \text{ then } 4w - 2T = 0$$

$$\Rightarrow 4w = 2T \quad \Rightarrow 2w = T \quad \text{or } T = 2w.$$

Hence, the tension (T) of the string is equal to half the weight of the four rods □.

(2a) To obtain the tangential component of velocity of a particle moving along a plane curve we consider a moving particle along a curve C. Let at time t and $(t + \partial t)$ its' position be at P and Q respectively, so that $\vec{OP} = \vec{r}$, $\vec{OQ} = \vec{r} + \partial\vec{r}$ fixed Origin.



Then, the velocity of P is

$$\vec{V} = \lim_{\partial t \to 0} \frac{(\vec{r} + \partial\vec{r}) - r}{\partial t} = \lim_{\partial t \to 0} \frac{\partial\vec{r}}{\partial t} = \frac{d\vec{r}}{dt} \quad \text{—— (i)}$$

Let A be a fixed point on the curve then $AP = S$ such that Arc $AQ = S + \partial S$, so that arc $PQ = \partial s$.

Then, $\frac{\partial\vec{r}}{\partial s} = \lim_{\partial s \to 0} \frac{\partial\vec{r}}{\partial s}$.

Here, $\partial \vec{r}$ is a vector in the direction of P to Q and when $Q \to P$, $\vec{PQ} = \partial \vec{r}$ tends to that of the tangent at P.

Let $\hat{\gamma}$ denote a unit velocity in the direction of the tangent, then, we may write
$$\frac{\partial \vec{r}}{\partial s} = \left| \lim_{\partial s \to 0} \frac{\partial \vec{r}}{\partial s} \right| \cdot \hat{\gamma}$$

But
$$\left| \lim_{\partial s \to 0} \frac{\partial \vec{r}}{\partial s} \right| = \left| \lim_{\partial s \to 0} \frac{\partial \vec{r}}{\partial s} \right| = \lim_{\partial s \to 0} \frac{|\partial \vec{s}|}{\partial s} = \lim_{Q \to P} \frac{\text{Chord } PQ}{\text{arc } PQ} = 1$$

This show that $\partial \vec{r}$ has unit magnitude and $\partial \vec{s}$ direction $\hat{T}$. Thus,
$$\frac{\partial \vec{r}}{\partial s} = \hat{T} \quad \text{——— (2)}$$

From (1),
$$\vec{V} = \frac{\partial \vec{r}}{\partial t} = \frac{\partial \vec{r}}{\partial s} \cdot \frac{\partial \vec{s}}{\partial t} = \frac{\partial \vec{s}}{\partial t} \hat{T}$$
$$\Rightarrow \vec{V} = \dot{S}\hat{T}.$$

Which shows that the velocity of the particle is along the tangent vector to its path and is of magnitude $\dot{S}$ □.

(2b) Given that Radial velocity ∝ Transverse Velocity.

i.e
$$\frac{dr}{dt} \propto r\frac{d\theta}{dt} \Rightarrow \frac{dr}{dt} = \lambda r \frac{d\theta}{dt} \Rightarrow dr = \lambda r d\theta \Rightarrow \frac{dr}{r} = \lambda d\theta.$$

where λ is the constant of proportionality. By integrating bothsides, we have,
$$\int \frac{dr}{r} = \lambda \int d\theta \Rightarrow \ln r = \lambda\theta + \ln k$$
where k is a constant of integration.
$$\Rightarrow \ln r - \ln k = \lambda\theta \Rightarrow \ln(r/k) = \lambda\theta$$
$$\Rightarrow \frac{r}{k} = e^{\lambda\theta} \Rightarrow r = ke^{\lambda\theta}.$$

Hence, the required path is an equiangular spiral □.

(3a) Consider a rigid body which turns about a point O fixed in both body and space. Let I_1, I_2, I_3 stands for the principle moments of inertia I and $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity $\vec{\omega}$ at any time t referred to the principle axes 1, 2, 3 which are essentially mutually orthogonal and fixed in the body. If $\vec{i}, \vec{j}, \vec{k}$ stipulate the unit vectors along these axes then the moments of momentum about O is instantaneously
$$\vec{L} = I_1\omega_1\vec{i} + I_2\omega_2\vec{j} + I_3\omega_3\vec{k} \quad \text{——— (1)}$$

The motion of a rigid body with one point fixed will take place under the action of a torque $\vec{N}$ in such a way that its total angular momentum varies at the rate equal to $\vec{N}$:
$$\frac{d\vec{L}}{dt} = \vec{N} \quad \text{——— (2)}$$

Here, the time derivative refers to the space axes and it holds only in an inertial system. In a coordinate system rotating between the two time derivatives;

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times \quad \text{--- (3)}$$

Equation (2) in terms of body axes is therefore;

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \vec{N} \quad \text{--- (4)}$$

In the case of eqn (1), we rember that the principal moments of inertia and the body base vectors $\vec{i}, \vec{j}, \vec{k}$ are constant in time with respect to the time derivative of $\vec{L}$ i.e $\frac{d\vec{L}}{dt}$ in rotating system'11,

$$\left(\frac{d\vec{L}}{dt}\right)_{body} = I_1\dot{\omega}_1\vec{i} + I_2\dot{\omega}_2\vec{j} + I_3\dot{\omega}_3\vec{k} \quad \text{--- (5)} \Rightarrow \vec{\omega}\times\vec{L} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$= (I_3 - I_2)\omega_2\omega_3\vec{i} - (I_3 - I_1)\omega_3\omega_1\vec{j} + (I_2 - I_1)\omega_1\omega_2\vec{k} \quad \text{--- (6)}$$

and $\vec{N} = N_1\vec{i} + N_2\vec{j} + N_3\vec{k}$ and using (5) & (6) in (4) we have; $N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3$, $N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1$

$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2$. --- (7) Known as the Euler's dynamical Equations.

If the resultant moment of the external forces is zero, then eqn (7) becomes;

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0,\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0,\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \quad \text{--- (8)}$$

Multiplying eqn (8) by $\omega_1, \omega_2, \omega_3$ and adding, we have;

$I_1\dot{\omega}_1\omega_1 + I_2\dot{\omega}_2\omega_2 + I_3\dot{\omega}_3\omega_3 = 0$. On integrating we have;

$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \text{Constant}$ (9) $\Rightarrow 2T = \text{Constant}$.

Where T is the kinetic energy during the motion.

(3b) From Euler's eqn. of motion Under no external forces, we have;

$$\left.\begin{aligned} I_1\dot{\omega}_1 - (I_1 - I_3)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 \end{aligned}\right\} \text{--- (1)}$$

Since $I_1 = I_2 = I_3$ for a cube, eqn (1) becomes

$$I_1\dot{\omega}_1 = 0,\ I_2\dot{\omega}_2 = 0 \text{ and } I_3\dot{\omega}_3 = 0$$

By integration, we have;

$$I_1\omega_1 = k_1,\ I_2\omega_2 = k_2 \text{ and } I_3\omega_3 = k_3$$

$$\Rightarrow \omega_1 = \frac{k_1}{I_1},\ \omega_2 = \frac{k_2}{I_2} \text{ and } \omega_3 = \frac{k_3}{I_3}.$$

Since I_1, I_2 and I_3 are all constant, then;

$$\omega_1 + \omega_2 + \omega_3 = \frac{k_1}{I_1} + \frac{k_2}{I_2} + \frac{k_3}{I_3}, \text{ a Constant.}$$

Also, $\omega_1^2 + \omega_2^2 + \omega_3^2 = \left(\frac{k_1}{I_1}\right)^2 + \left(\frac{k_2}{I_2}\right)^2 + \left(\frac{k_3}{I_3}\right)^2$ which is also a constant.

(4) Consider a dynamical system consisting of N particles each with m_i and position vector $\vec{r}_i$ where $i = 1, 2 \cdots N$. Let the motion of the system at any point be described by n generalized coordinates q_i where $i = 1, 2 \cdots n$. Then $\vec{r}_i$ can be expressed in terms of the generalized coordinates q_i and possibly time t.

$$\vec{r}_i = \vec{r}_i(q_i, t) = \vec{r}_i(q_1, q_2, \cdots q_n, t) \quad \text{--- (1)}$$

By differentiating with respect to time t, the velocity can be written as,

$$\dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}$$

$$= \sum_{i=1}^{n} \frac{\partial \vec{r}_i}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_i}{\partial t} = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \quad \text{--- (2)}$$

Now, in order to get K.E, one has to square eqn (2). This will give three types of terms; the term involving square of the velocity, the term involving the velocity and the term independent of the velocity.

$$\Rightarrow \dot{\vec{r}}_i^{\,2} = \sum_{j,k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k + 2\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j \cdot \frac{\partial \vec{r}_i}{\partial t} + \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 \quad \text{--- (3)}$$

But the K.E, $T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$.

$$\therefore \cancel{T = \sum_{j,k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}}$$

$$\therefore T = \sum_{j,k=1}^{n} a_{jk}\dot{q}_j\dot{q}_k + \sum_{j=1}^{n} b_j \dot{q}_j + c$$

where $a_{jk} = \frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$,

$$b_j = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t}$$

$$c = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2.$$

Which is the general expression for Kinetic energy of a dynamic system.

5a) The position of the bead includes $x = a(1-\sin\theta)$ and $y = a(1+\cos\theta)$ so that velocity components includes. $\dot{x} = -a\cos\theta\dot{\theta}$ and $\dot{y} = -a\sin\theta\dot{\theta}$

Such that $v^2 = \dot{x}^2 + \dot{y}^2 = (-a\cos\theta\dot{\theta})^2 + (-a\sin\theta\dot{\theta})^2$

$$= a^2\cos^2\theta\dot{\theta}^2 + a^2\sin^2\theta\dot{\theta}^2$$

$$= a^2(\cos^2\theta + \sin^2\theta)\dot{\theta}^2 = a^2\dot{\theta}^2$$

Kinetic energy interms of $\dot{\theta}$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\dot{\theta}^2$$

If it slides frictionlessly under gravity from rest at the top i.e $\theta = 0$

Using the law of Conservation of energy with height $y = a(1+\cos\theta)$, $T = 0$ at $\theta = 0$

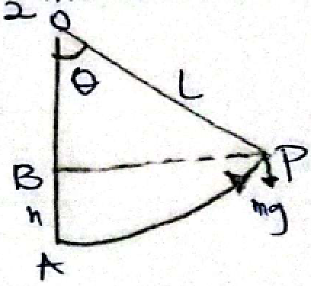
$$T(\theta) = mg(y(0) - y(\theta)] = mga(1-\cos\theta)$$

$$\dot{\theta} = 2\sqrt{g/a}\sin\frac{\theta}{2}, \quad V = a\dot{\theta} = 2\sqrt{ga}\sin\frac{\theta}{2}$$

$$\Rightarrow T = 2mag\sin^2\frac{\theta}{2}$$

5b) Let m be the mass of the body of a simple pendulum whos strip is of length C. Let θ be the angle between the rest position and the deflected position. The K.E is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$



In coming from position P to A, the mass has fallen freely through a vertical distance AB. Then the potential energy $V = mgh = mgAB$

$$= mg(OA - OB) = mg(L - L\cos\theta)$$

$$= mgL(1-\cos\theta).$$

By Lagrangian, $L = T - V$ = Kinetic − potential.

$$= \frac{1}{2}mL^2\dot{\theta}^2 - mgL(1-\cos\theta)$$

By Lagrangian's equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(ml^2\dot{\theta}) - (-mgL\sin\theta) = \frac{d}{dt}(ml^2\dot{\theta}) + mgL\sin\theta$$

$$= ml^2\ddot{\theta} + mgl\sin\theta = 0.$$

If θ is very small, then $\sin\theta \simeq \theta$

$$\therefore ml^2\ddot{\theta} + mgl\theta = 0 \text{ or } l^2\ddot{\theta} + gl\theta = 0 \text{ or } \ddot{\theta} + \frac{g}{l}\theta = 0$$

If $\frac{g}{L} = \mu \Rightarrow \ddot{\theta} + \mu\theta = 0$

This is the equation of motion of a simple pendulum which shows that the motion of a simple pendulum is harmonic or simple harmonic motion.

6. The law of conservation of angular momentum states that if the sum of the external torques about a fixed point acting on a system of particle is zero, then the angular momentum of the system about the fixed point is constant throughout the motion.

Proof: Consider a mass-system S consisting of n-particles P_i each with a constant mass m_i and position vector $\vec{r}_i$ with respect to the origin O of an inertia frame. Then by Newton's second law, the equation of motion of the particle P_i can be written as, $\frac{d}{dt}(m_i\dot{\vec{r}}_i) = \vec{F}_i + \sum_{k=1}^{n}\vec{F}_{ik}$ — (1)

Where $\vec{F}_i$ is the total external forces acting on the particle P_i and $\vec{F}_{ik}$ is the internal forces which the particle P_k exerts on the particle P_i.

To obtain an expression for the rate of change of the angular momentum, we take the vector product of (1) with the position vector $\vec{r}_i$, we get $\vec{r}_i \times \frac{d}{dt}(m_i\dot{\vec{r}}_i) = \vec{r}_i \times \vec{F}_i + \sum_{k=1}^{n}\vec{r}_i \times \vec{F}_{ik}$ — (2)

The angular momentum of the particle P_i is $\vec{H}_i = \vec{r}_i \times m_i\dot{\vec{r}}_i$

Now, $\frac{d\vec{H}_i}{dt} = \frac{d}{dt}(\vec{r}_i \times m_i\dot{\vec{r}}_i) = \vec{r}_i \times \frac{d}{dt}(m_i\dot{\vec{r}}_i) + \dot{\vec{r}}_i \times m_i\dot{\vec{r}}_i$

$$= \vec{r}_i \times \frac{d}{dt}(m_i\dot{\vec{r}}_i) \quad \text{— (3)}$$

From eqn (2) and (3), we have;

$$\frac{d\vec{H}_i}{dt} = \vec{r}_i \times \vec{F}_i + \sum_{k=1}^{n}\vec{r}_i \times \vec{F}_{ik}$$

Summing over all particles, we have,

$$\frac{d\vec{H}}{dt} = \sum_{i=1}^{n} \frac{d\vec{H}_i}{dt} = \sum_{i=1}^{n} \vec{r}_i \times \vec{f}_i + \sum_{i=1}^{n}\sum_{k=1}^{n} \vec{r}_i \times \vec{f}_{ik} \quad \text{--- (4)}$$

Since the internal forces are central and the torques due to the interacting particles are oppositely directed by equal in magnitude

Hence,

$$\sum_{i=1}^{n}\sum_{k=1}^{n} \vec{r}_i \times \vec{f}_{ik} = 0.$$

Therefore, eqn (4) becomes

$$\frac{d\vec{H}}{dt} = \sum_{i=1}^{n} \vec{r}_i \times \vec{f}_i \quad \text{--- (5)}$$

If the summing of the external torques about a fixed point acting on the system is zero, i.e

$$\sum_{i=1}^{n} \vec{r}_i \times \vec{f}_i = 0.$$

Then, eqn (5) reduces to

$$\frac{d\vec{H}}{dt} = 0$$

D.